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Critical properties of Dyson's hierarchical model

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Abstract. Critical properties of Dyson's hierarchical Ising model in one dimension with 'potential' falling off like $r^{-(1+\sigma)}$ are examined in the range $0<\sigma<1$ where a phase transition is known to occur. A new exact renormalisation group recursion relation is derived for a 'dual' spin probability density function. Together with a scaling-type assumption we obtain $\eta=2-\sigma$ and $\delta=(1+\sigma)/(1-\sigma)$ for all $0<\sigma<1$. Independent evidence, however, suggests that $\delta=3$ for $0<\sigma\leqslant 1/2$, along with other classical critical exponents in this region. In the non-classical region $1/2<\sigma<1$ we obtain accurate numerical values for the critical exponents ν and γ and various $\Delta\sigma=\sigma-1/2$ expansions to third order in $\Delta\sigma$. The latter are in close agreement with our numerical estimates for small $\Delta\sigma$ but are in disagreement with similar expansions obtained to second order by Blekher and Sinai. The numerical values for $1/2<\sigma<1$ seem to fit no simple formula and suggest non-analytic behaviour of critical exponents as $\sigma\to 1_-$.

1. Introduction

Dyson's hierarchical model (Dyson 1969, 1971), to be referred to as HM, is a lattice model which simulates a system with power-law long-range interactions. It was originally proposed by Dyson (1969) to prove the existence of a phase transition for a one-dimensional Ising model with a power-law potential. Subsequently it was realised (Baker 1972) that the HM provides an example to which Wilson's renormalisation group ideas (Wilson 1971) can be applied exactly. Blekher and Sinai (1974) also have announced a detailed investigation of the HM whose potential falls off like $r^{-d\xi}$ as $r\to\infty$, d being the dimensionality of the system and $1<\xi<2$ to guarantee the existence of a phase transition. They show that the critical behaviour of the model is determined by a fixed point solution of a certain non-linear integral equation with a Gaussian solution as the stable physical solution associated with 'classical' critical exponents for $1<\xi<3/2$, while for $2>\xi>3/2$ a non-Gaussian solution, resulting in non-classical behaviour, is the appropriate physical solution. They also obtained expansions for various critical exponents to second order in $\xi-3/2>0$ for $d=1$ which are similar to the usual $\epsilon=4-d$ expansions in renormalisation group calculations (Wilson and Kogut 1974). Similar results have been obtained by Fisher *et al* (1972) for the corresponding power-law potential Ising model, confirming the close similarity with the HM in the neighbourhood of the critical point. The spherical version of the HM has also been analysed (McGuire 1973) and shown to have identical critical behaviour to the corresponding power-law potential spherical model. A general class of models, termed asymptotically hierarchical, has also been considered by Blekher and Sinai (1973, 1975) with particular attention given to the existence of fixed points.

Our purpose here is to further investigate the critical behaviour of the HM in one dimension with potential falling off like $r^{-(1+\sigma)}$. While our approach parallels that of Blekher and Sinai (1974) and also Baker (1972) we follow a slightly different procedure that results in a new renormalisation group recursion formula that is ideally suited to analytical and numerical analysis. In § 3 the critical exponents δ and η, which govern the behaviour of the system at the critical point, are determined assuming that at criticality certain functions defined recursively by the renormalisation group transformation possess limits. Our results here are in essence in agreement with Blekher and Sinai (1974), Baker (1972) and Baker and Golner (1973, 1977). There is some doubt, however, about the validity of our assumptions, particularly concerning δ, in the region $0<\sigma<1/2$ where independent numerical analysis of series (Guttmann *et al* 1977) suggests that δ sticks at three in this region. In § 4 we investigate the renormalisation procedure analytically in the region $0<\sigma\leqslant 1/2$ and numerically over the whole range of $0<\sigma<1$, obtaining all relevant critical exponents to high accuracy. In § 5 we develop expansions in powers of $\Delta\sigma=\sigma-1/2$ to third order in the non-classical regime $1>\sigma>1/2$. Our $\Delta\sigma$ expansions differ from those of Blekher and Sinai (1974) in the second-order term, but agree with our numerical results for small $\Delta\sigma$. The numerical results of Blekher (Blekher and Sinai 1975) and most resently of Baker and Golner (1977) are, however, in qualitative agreement with ours. Finally, we conclude in § 6 with a discussion and summary of our results.

2. Exact renormalisation of the hierarchical model

The Hamiltonian of the N-level, 2^N-spin, HM we consider here is given by (Dyson 1969)

$$\mathscr{H}_N=-\sum_{p=1}^{N}2^{-2p}c^p\sum_{r=1}^{2^{N-p}}(S_{p,r})^2-HS_{N,1} \tag{2.1}$$

where $S_{p,r}$ is the spin-sum of 2^p Ising spins in the rth block at the pth level, i.e.

$$S_{p,r}=S_{p-1,2r-1}+S_{p-1,2r}=\sum_i\mu_i, \qquad \text{for } (r-1)2^p+1\leqslant i\leqslant r2^p, \tag{2.2}$$

$\mu_i=\pm 1$, $c=2^{1-\sigma}$ and H is the reduced magnetic field.

This model is expected to behave like the one-dimensional Ising model with potential $J_{ij}=|i-j|^{-(1+\sigma)}$. In fact Dyson (1969) has proved that in the thermodynamic limit, (2.1), and as a consequence the Ising model with potential J_{ij} above, has a phase transition when $0<\sigma<1$. In the sequel we restrict our consideration to this range of σ.

Starting from (2.1) with $N=l$ and separating off the term $p=l$, the partition function for 2^l-spins can be written as

$$\begin{aligned}Q_l^{(0)}(\beta,H)&=\sum_{\{\mu_i=\pm1\}}\exp(-\beta\mathscr{H}_l)\\&=\sum_{\{\mu_i=\pm1\}}\exp\left(\beta\sum_{p=1}^{l-1}2^{-2p}c^p\sum_{r=1}^{2^{l-p}}(S_{p,r})^2+\beta HS_{l,1}+\beta 2^{-2l}c^l(S_{l,1})^2\right).\end{aligned} \tag{2.3}$$

Using the elementary identity

$$e^{ay^2}=\pi^{-1/2}\int_{-\infty}^{\infty}e^{-x^2+2a^{1/2}xy}\,dx \tag{2.4}$$

with $a = \beta 2^{-2l} c^l$ and $y = S_{l,1}$, together with the hierarchical structure of the model (in particular $S_{l,1} = S_{l-1,1} + S_{l-1,2}$) we immediately obtain (Thompson 1972),

$$Q_l^{(0)}(\beta, H) = \pi^{-1/2} \int_{-\infty}^{\infty} e^{-x^2} [Q_{l-1}^{(0)}(\beta, H + \beta^{-1/2} 2^{1-l} c^{l/2} x)]^2 \, dx. \tag{2.5}$$

In essence the transformation (2.4) decouples the top (lth) level of the hierarchy and splits the model into two equivalent 2^{l-1}-spin hierarchical models with a modified field.

Defining $\tilde{P}_l(x)$ by

$$Q_l^{(0)}(\beta, H) = \{\tilde{P}_{l+1}[(4^l \beta / c^{l+1})^{1/2} H]\}^{1/2} \tag{2.6}$$

equation (2.5) can be written as

$$\tilde{P}_{l+1}[(4^l \beta / c^{l+1})^{1/2} H] = \left(\pi^{-1/2} \int_{-\infty}^{\infty} e^{-x'^2} \tilde{P}_l[(4^{l-1} \beta / c^l)^{1/2} (H + (c^l / 4^{l-1} \beta)^{1/2} x')] \, dx'\right)^2 \tag{2.7}$$

changing variables to $H = (c^l / 4^{l-1} \beta)^{1/2} y$ and $x = y + x'$ then gives

$$\tilde{P}_{l+1}(2c^{-1/2} y) = \left(\pi^{-1/2} \int_{-\infty}^{\infty} e^{-(x-y)^2} \tilde{P}_l(x) \, dx\right)^2 \tag{2.8}$$

which is our basic renormalisation group recursion relation.

In previous treatments of the HM, renormalisation group recursion relations have usually been derived for spin probability distribution functions. In particular, following Baker (1972), one can begin with the Ising distribution function

$$P_0(x) = \delta(x+1) + \delta(x-1) \tag{2.9}$$

and apply the standard renormalisation group recipe (Wegner 1972) as follows.

(*a*) *Dilation*

First extend the 2^N-spin system by a factor of 2 and write

$$Q_{N+1}^{(0)}(\beta, H) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \exp(-\beta \mathcal{H}_{N+1}) \prod_{r=1}^{2N+1} P_0(S_{0,r}) \, dS_{0,r}. \tag{2.10}$$

(*b*) *Contraction by a partial trace*

Of the 2^{N+1} degrees of freedom in (2.10) we eliminate half of them by a partial trace, which can be performed exactly in this case due to the hierarchical structure of the model. The details can be found in Baker (1972).

(*c*) *Spin scaling and relabelling*

To regain the interaction term for the 2^N-spin model we scale the spins by a factor $c^{1/2}/2$ and relabel by writing

$$S'_{p-1,r} = c^{1/2} S_{p,r} / 2. \tag{2.11}$$

Repeating this process l times we define

$$Q_N^{(l)}(\beta, H) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \exp(-\beta \mathcal{H}_N) \prod_{r=1}^{2N} P_l(S_{0,r}) \, dS_{0,r} \tag{2.12}$$

and

$$-\beta f^{(l)}(\beta, H) = \lim_{N\to\infty} 2^{-N} \ln Q_N^{(l)}(\beta, H). \tag{2.13}$$

$f^{(0)}(\beta, H)$ is the free energy of the system in the thermodynamic limit and $P_l(x)$ is defined recursively by

$$P_{l+1}(c^{1/2}y) = 2c^{-1/2}\exp(\beta c y^2)\int_{-\infty}^{\infty} P_l(y+x)P_l(y-x)\,dx. \tag{2.14}$$

The renormalisation process then states that

$$f^{(0)}(\beta, H) = 2^{-l} f^{(l)}(\beta, 2^l c^{-l/2} H) \tag{2.15}$$

for all β, H and l. We also note that

$$Q_N^{(0)}(\beta, H) = Q_0^{(N)}(\beta, 2^N c^{-N/2} H) = \int_{-\infty}^{\infty} \exp(\beta 2^N c^{-N/2} H y) P_N(y)\,dy. \tag{2.16}$$

It is an interesting property of the HM that one can construct a 'dual' model Hamiltonian $\mathscr{H}'_N$, also with hierarchical structure, such that $Q_N^{(l)}(\beta, H)$ defined by (2.12) can be written as

$$Q_N^{(l)}(\beta, H) = \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} \exp(-\beta\mathscr{H}'_N)\prod_{r=1}^{2N}\tilde{P}_l(S_{0,r})\,dS_{0,r} \tag{2.17}$$

with $\tilde{P}_l(x)$ and $P_l(x)$ related by

$$\tilde{P}_l(x) = \int_{-\infty}^{\infty} \exp(-\beta y^2 + 2\beta^{1/2} y x) P_l(y)\,dy. \tag{2.18}$$

The details are given in the appendix. The important thing to note here is the obvious interpretation of the $\tilde{P}_l(x)$ as (dual) spin distribution functions.

The starting point for our recursion relation, from (2.9) and (2.18), is

$$\tilde{P}_0(x) = 2\,e^{-\beta}\cosh(2\beta^{1/2}x), \tag{2.19}$$

which although not defined by (2.6), gives from (2.8), the required $Q_N^{(0)}(\beta, H)$ specified by (2.6) and (2.17). The details are also given in the appendix.

Finally we note that even though the recursion relation (2.8) has no explicit β dependence, the $\tilde{P}_l(x)$ are all temperature dependent through (2.19). This temperature dependence, although suppressed, should be kept in mind in the sequel.

3. Critical exponents δ and η

In essence, the renormalisation group scheme states that at $\beta = \beta_c$, a normalised spin distribution function, e.g.

$$\pi_l(x) = \tilde{P}_l(x)/\tilde{P}_l(0) \tag{3.1}$$

approaches a limit as $l \to \infty$, and that the critical behaviour of the system is determined by the properties of that particular fixed point of the renormalisation group equation, which in our case is (2.8).

For the HM we are able to determine the values of δ and η which govern the critical behaviour of the system at $\beta=\beta_c$, *assuming that at $\beta=\beta_c$, the 'free energy' $\bar{f}^{(l)}(\beta_c, H)$, and the pair correlation function $\Gamma^{(l)}(r, r')$, defined below, approach limits as $l\to\infty$.* Granted this assumption, which we have been unable to prove, δ and η are determined regardless of the particular fixed point of (2.8) the system approaches. While this is in agreement with results obtained from renormalisation group analysis of the power law long-range potential n-vector model (Fisher *et al* 1972, see also Baker 1972) it is a little disturbing that in the so called 'classical region' $0<\sigma<1/2$, our δ does not take its classical value of three, as it does for example for the hierarchical spherical model (McGuire 1973). It may well be that our assumption breaks down in this region (at least for $\bar{f}^{(l)}(\beta_c, H)$) but we have been unable to resolve this question at the present time except to note that if the Gaussian fixed point ($0<\sigma<1/2$) is used in place of $\pi_l(x)$ below, the resulting 'free energy' is not defined. In addition, preliminary numerical study of series expansions (Guttmann *et al* 1977) does indeed suggest that δ sticks at three for $0<\sigma<1/2$. A similar difficulty, which seems to be studiously avoided, arises in renormalisation group treatments of short-range interaction d-dimensional models, where the question of what value δ takes in the classical region $d>4$, is, as far as we are aware, unresolved. Finally it should be mentioned that similar points for the HM have been raised by Baker and Golner (1973). In the remainder of this section we will operate under the above assumption.

In terms of $\pi_l(x)$, (2.12), using the relations involving $\tilde{P}_l(x)$ and $P_l(x)$ of the previous section, can be written as

$$Q_N^{(l)}(\beta, H)=[\tilde{P}_l(0)]^{2N}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\exp(-\beta\mathscr{H}'_N)\prod_{r=1}^{2N}\pi_l(S_{0,r})\,\mathrm{d}S_{0,r}. \qquad (3.2)$$

Defining

$$\bar{Q}_N^{(l)}(\beta, H)=\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\exp(-\beta\mathscr{H}'_N)\prod_{r=1}^{2N}\pi_l(S_{0,r})\,\mathrm{d}S_{0,r} \qquad (3.3)$$

and

$$-\beta\bar{f}^{(l)}(\beta, H)=\lim_{N\to\infty}2^{-N}\ln\bar{Q}_N^{(l)}(\beta, H) \qquad (3.4)$$

and noting that $\tilde{P}_l(0)$ does not depend on H, we have

$$f^{(l)}(\beta, H)-f^{(l)}(\beta, 0)=\bar{f}^{(l)}(\beta, H)-\bar{f}^{(l)}(\beta, 0) \qquad (3.5)$$

where $f^{(l)}(\beta, H)$ is defined by (2.13). It then follows from (2.15) that

$$\begin{aligned} f^{(0)}(\beta, H)-f^{(0)}(\beta, 0) &\\ &=2^{-l}[f^{(l)}(\beta, 2^l c^{-l/2}H)-f^{(l)}(\beta, 0)]\\ &=2^{-l}[\bar{f}^{(l)}(\beta, 2^l c^{-l/2}H)-\bar{f}^{(l)}(\beta, 0)]. \end{aligned} \qquad (3.6)$$

Now, by definition

$$f^{(0)}(\beta_c, H)-f^{(0)}(\beta_c, 0)\sim H^{1+1/\delta} \qquad \text{as } H\to 0+. \qquad (3.7)$$

Consequently, since $2^{-l}c^{l/2}=2^{-l(1+\sigma)/2}$, we have for *fixed H* and $l\to\infty$,

$$\begin{aligned}\bar{f}^{(l)}(\beta_c,H)-\bar{f}^{(l)}(\beta_c,0)&\\ &=2^l[f^{(0)}(\beta_c,2^{-l}c^{l/2}H)-f^{(0)}(\beta_c,0)]\sim 2^l2^{-l(1+1/\delta)}c^{l(1+1/\delta)/2}H^{1+1/\delta}\\ &=2^{l[(1-\sigma)-(1+\sigma)/\delta]/2}H^{1+1/\delta}.\end{aligned}\tag{3.8}$$

So far our analysis is exact. If we now assume that the left-hand side of (3.8) has a limit as $l\to\infty$ it follows immediately that

$$(1-\sigma)-(1+\sigma)/\delta=0$$

and hence

$$\delta=(1+\sigma)/(1-\sigma).\tag{3.9}$$

We note that when $\sigma=1/2$, (3.9) gives $\delta=3$ and that the only way this value can be maintained for $0<\sigma<1/2$ is for the left-hand side of (3.8) to diverge as $l\to\infty$.

To derive an expression for η we first define the spin–spin correlation function $\Gamma_l(r,r')$ with respect to the probability density $P_l(x)$ *at criticality* by

$$\begin{aligned}\Gamma_l(r,r')&=\langle S_{0,r}S_{0,r'}\rangle_l\\ &=\Big(\lim_{N\to\infty}\Big)\Big(\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}S_{0,r}S_{0,r'}\exp[-\beta_c\mathscr{H}_N(H=0)]\\ &\qquad\times\prod_{r=1}^{2N}P_l(S_{0,r})\,dS_{0,r}/Q_N^{(l)}(\beta_c,0)\Big).\end{aligned}\tag{3.10}$$

Next we consider

$$\begin{aligned}\langle S_{1,r}S_{1,r'}\rangle_l=\lim_{N\to\infty}\Big(\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}&S_{1,r}S_{1,r'}\exp[-\beta_c\mathscr{H}_N(H=0)]\\ &\times\prod_{r=1}^{2N}P_l(S_{0,r})\,dS_{0,r}/Q_N^{(l)}(\beta_c,0)\Big).\end{aligned}\tag{3.11}$$

Applying the renormalisation process (*a*), (*b*), (*c*) described in the previous section to (3.11), $P_l(x)$ goes to $P_{l+1}(x)$, $\mathscr{H}_N(H=0)$ and $Q_N^{(l)}(\beta_c,0)$ are unchanged, and $S_{1,r}S_{1,r'}$ goes to $2^{1+\sigma}S_{0,r}S_{0,r'}$. Hence

$$\langle S_{1,r}S_{1,r'}\rangle_l=2^{1+\sigma}\langle S_{0,r}S_{0,r'}\rangle_{l+1}=2^{1+\sigma}\Gamma_{l+1}(r,r').\tag{3.12}$$

On the other hand,

$$\begin{aligned}\langle S_{1,r}S_{1,r'}\rangle_l&=\langle(S_{0,2r-1}+S_{0,2r})(S_{0,2r'-1}+S_{0,2r'})\rangle_l\\ &=\Gamma_l(2r-1,2r'-1)+\Gamma_l(2r,2r'-1)+\Gamma_l(2r-1,2r')+\Gamma_l(2r,2r').\end{aligned}\tag{3.13}$$

However, as shown by Dyson (1969), all four terms on the right-hand side of (3.13) are equal when $r\neq r'$, so that on combining (3.12) and (3.13) we have

$$2^{1+\sigma}\Gamma_{l+1}(r,r')=4\Gamma_l(2r,2r')\tag{3.14}$$

and hence by iteration

$$\Gamma_l(r,r')=2^{-l(\sigma-1)}\Gamma_0(2^lr,2^lr').\tag{3.15}$$

Now, by definition, since the dimensionality d is unity,

$$\Gamma_0(r,r')\sim|r-r'|^{-(d-2+\eta)}=|r-r'|^{-(\eta-1)}\qquad\text{as }|r-r'|\to\infty.\tag{3.16}$$

Equations (3.16) and (3.15) then imply that for fixed $r \neq r'$ and $l \to \infty$

$$\Gamma_l(r, r') \sim 2^{l(2-\sigma-\eta)}|r - r'|^{-(\eta-1)}. \tag{3.17}$$

Hence, assuming that $\Gamma_l(r, r')$ approaches a limit as $l \to \infty$, we deduce that

$$\eta = 2 - \sigma. \tag{3.18}$$

Our results (3.9) and (3.18) are in agreement with previous workers, and we note that these values for δ and η satisfy the scaling relation

$$d(\delta - 1)/(\delta + 1) = 2 - \eta \tag{3.19}$$

with $d = 1$. While it is likely that (3.18) is valid for the whole range $0 < \sigma < 1$, recent series analysis (Guttmann *et al* 1977) suggests that (3.9) is valid for $1 > \sigma \geq 1/2$ but that δ sticks at three for $0 < \sigma < 1/2$. If these latter results are valid the scaling relation (3.19) fails in the region $0 < \sigma < 1/2$, which is also the case for the HM spherical model.

4. Fixed point analysis

To calculate the critical exponents ν, γ etc which characterise the behaviour of the system as $\beta \to \beta_c$, we need to know, first of all, to which fixed point of the renormalisation group equation (RGE) (2.8) the system approaches *at* $\beta = \beta_c$. The critical exponents are then determined using well established arguments (Wilson and Kogut 1974), by linearisation around the appropriate fixed point. It should be stressed that a neighbourhood analysis of a fixed point alone is not enough to guarantee that the fixed point is the appropriate physical fixed point. One really needs to prove, as was done by Blekher and Sinai (1973) for the Gaussian fixed point of the asymptotically hierarchical models, that starting from $\tilde{P}_0(x)$ at criticality the RGE approaches the fixed point in question. We make no attempt here to discuss this problem rigorously. Rather, we content ourselves with a numerical determination of the appropriate fixed point and a neighbourhood analysis of that fixed point to determine the critical exponents.

A convenient basis to work with here consists of Hermite polynomials $H_k(ax)$, of degree k, with $|a| \leq 1$ to be fixed in a moment. This choice is motivated by the fact that

$$\pi^{-1/2} \int_{-\infty}^{\infty} \exp[-(x-y)^2] H_k(ax)\, dx = (1-a^2)^{k/2} H_k(a(1-a^2)^{-1/2} y). \tag{4.1}$$

Hence we expand $\tilde{P}_l(x)$ in terms of Hermite polynomials and for convenience choose a normalisation scheme so that the zeroth-order coefficient is always unity. That is, we write

$$\tilde{P}_l(x) = B_0^{(l)}\left(1 + \sum_{k=1}^{\infty} B_k^{(l)} 2^{(1+\sigma)k} H_{2k}(ax)\right). \tag{4.2}$$

The factor $2^{(1+\sigma)k}$ is introduced for later convenience, and only even orders of H_k are present since $\tilde{P}_l(x)$ is even. The constant a is chosen to match (4.1) and the left-hand side of the RGE (2.8), i.e.

$$a = (1 - 2^{-(1+\sigma)})^{1/2}. \tag{4.3}$$

We now express the RGE for $\tilde{P}_l(x)$ in terms of its Hermite coefficients $B_k^{(l)}$. First, using (4.1) we have

$$\pi^{-1/2} \int_{-\infty}^{\infty} \exp[-(x-y)^2] \tilde{P}_l(x)\, dx = B_0^{(l)}\left(1 + \sum_{k=1}^{\infty} B_k^{(l)} H_{2k}(2^{(1+\sigma)/2} ay)\right). \tag{4.4}$$

On squaring (4.4), the RGE (2.8) becomes

$$B_0^{(l+1)}\left(1+\sum_{k=1}^{\infty} B_k^{(l+1)}2^{(1+\sigma)k}H_{2k}(z)\right)$$

$$=(B_0^{(l)})^2\left(1+2\sum_{k=1}^{\infty} B_k^{(l)}H_{2k}(z)+\sum_{k'=1}^{\infty}\sum_{k''=1}^{\infty} B_{k'}^{(l)}B_{k''}^{(l)}H_{2k'}(z)H_{2k''}(z)\right) \tag{4.5}$$

where

$$z=2^{(1+\sigma)/2}ay. \tag{4.6}$$

The product of the two Hermite polynomials in (4.5) can be expressed as a linear combination of Hermite polynomials (Erdélyi *et al* 1953), reducing the double sum in (4.5) to

$$\sum_{k'=1}^{\infty}\sum_{k''=1}^{\infty} B_{k'}^{(l)}B_{k''}^{(l)}H_{2k'}(z)H_{2k''}(z)$$

$$=\sum_{k'=1}^{\infty}\sum_{k''=1}^{\infty} B_{k'}^{(l)}B_{k''}^{(l)}\sum_{k=|k'-k''|}^{k'+k''} 2^l l!\binom{2k'}{l}\binom{2k''}{l}H_{2k}(z) \tag{4.7}$$

where $l=k'+k''-k$. The right-hand side of (4.5) can then be written in the form

$$(B_0^{(l)})^2\left[1+\sum_{k=1}^{\infty} 2^{2k}(2k)!(B_k^{(l)})^2+\sum_{k=1}^{\infty}\left(2B_k^{(l)}+\sum_{k'=1}^{\infty}\sum_{k''=1}^{\infty} T_{k,k',k''}B_{k'}^{(l)}B_{k''}^{(l)}\right)H_{2k}(z)\right] \tag{4.8}$$

where

$$T_{k,k',k''}=\begin{cases} 2^l l!\binom{2k'}{l}\binom{2k''}{l} & \text{if } |k'-k''|\leqslant k\leqslant k'+k'' \\ 0 & \text{otherwise,}\end{cases} \tag{4.9}$$

is determined by the linearisation of the RGE (4.10)–(4.12) around the fixed point (4.13), namely,

$$\Delta B_k^{(l+1)} = B_k^{(l+1)} - B_k^* = \sum_{k'=1}^{\infty} V_{k,k'} \Delta B_{k'}^{(l)}, \qquad k = 1, 2, \ldots, \tag{4.14}$$

where $V_{k,k'}$ is a matrix whose elements are given by

$$V_{k,k'} = \frac{2}{\lambda^*}\left[2^{-(1+\sigma)k}\left(\delta_{kk'} + \sum_{k''=1}^{\infty} T_{k,k',k''} B_{k''}^*\right) - B_k^* B_{k'}^* 2^{2k'}(2k')!\right] \tag{4.15}$$

and

$$\lambda^* = 1 + \sum_{k=1}^{\infty} 2^{2k}(2k)!(B_k^*)^2. \tag{4.16}$$

In order to determine the appropriate physical fixed point, one must start from $\tilde{P}_0(x)$, given by (2.19), expressed in terms of its Hermite expansion (4.2). Thus, using the formula (Erdélyi *et al* 1953)

$$\exp(2xz - z^2) = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x) \tag{4.17}$$

we obtain

$$\tilde{P}_0(x) = 2\exp\left(-\beta + \frac{\beta}{a^2}\right)\left[1 + \sum_{k=1}^{\infty}\left(\frac{\beta}{a^2}\right)^k \frac{H_{2k}(ax)}{(2k)!}\right]. \tag{4.18}$$

Comparing (4.2) and (4.18) we obtain

$$B_k^{(0)} = [\beta/(2^{1+\sigma} - 1)]^k/(2k)! \tag{4.19}$$

as the starting point for the recursion. Thus for given σ and β, $B_k^{(l)}$ for any k and l are generated by equations (4.19), (4.11) and (4.12). Further, by adjusting β, one finds $\beta = \beta_c$ so that $B_k^{(l)} \to B_k^*$ as $l \to \infty$. Finally, once the appropriate fixed point is known, the critical exponents are obtained using well established arguments (Wilson and Kogut 1974) from the maximum eigenvalue Λ_1 of the matrix $V_{k,k'}$. For example

$$\nu = \ln 2/\ln \Lambda_1, \tag{4.20}$$

$$\gamma = \sigma \ln 2/\ln \Lambda_1, \tag{4.21}$$

etc, with other exponents obtained from scaling laws, subject to the requirement that $\Lambda_1 > 1$ and all other eigenvalues of $V_{k,k'}$ are less than unity in absolute value.

An obvious candidate for a fixed point is the Gaussian solution

$$\tilde{P}_l(x) = \tilde{P}^*(x) = \exp[(1 - 2^{-\sigma})x^2] \tag{4.22}$$

of the RGE (2.8). In this case the Hermite coefficients B_k^* defined by (4.2) are easily found to be

$$B_k^* = [(1 - 2^{-\sigma})/4]^k/k!, \qquad k = 1, 2, \ldots. \tag{4.23}$$

Our numerical procedure described below in fact confirms that this is the appropriate fixed point when $0 < \sigma < 1/2$, in agreement with Blekher and Sinai (1974). The prescription above for determining critical exponents, or equivalently the maximum eigenvalue Λ_1 of the linearised RGE (4.14), is in this case, however, rather unwieldy. We found it more convenient to separate off the Gaussian part by writing

$$\tilde{P}_l(x) = \exp[(1-2^{-\sigma})x^2]h_l(2^{-\sigma/2}x) \tag{4.24}$$

and expanding $h_l(x)$ in terms of Hermite polynomials. That is, if we write $(c = 2^{1-\sigma})$

$$h_l(x) = A_0^{(l)}\left(1 + \sum_{k=1}^{\infty} A_k^{(l)} c^k H_{2k}(a'x)\right) \tag{4.25}$$

and follow the steps above leading to (4.10)–(4.12), we obtain (with $a' = (1-c^{-1})^{1/2}$),

$$A_0^{(l+1)} = 2^{\sigma}(A_0^{(l)})^2\lambda' \tag{4.26}$$

$$\lambda' c^k A_k^{(l+1)} = 2A_k^{(l)} + \sum_{k'=1}^{\infty}\sum_{k''=1}^{\infty} T_{k,k',k''}A_{k'}^{(l)}A_{k''}^{(l)}, \qquad k = 1, 2, \ldots, \tag{4.27}$$

and

$$\lambda' = 1 + \sum_{k=1}^{\infty} 2^{2k}(2k)!(A_k^{(l)})^2, \tag{4.28}$$

where $T_{k,k',k''}$ is given by (4.9). Also, in terms of a general fixed point A_k^* of (4.27)–(4.28) the linearised equations are

$$\Delta A_k^{(l+1)} = A_k^{(l+1)} - A_k^* = \sum_{k'=1}^{\infty} V'_{k,k'}\Delta A_{k'}^{(l)}, \qquad k = 1, 2, \ldots, \tag{4.29}$$

where

$$V'_{k,k'} = \frac{2}{\lambda'^*}\left(c^{-k}\delta_{kk'} + c^{-k}\sum_{k''=1}^{\infty} T_{k,k',k''}A_{k''}^* - 2^{2k'}(2k')!A_k^*A_{k'}^*\right). \tag{4.30}$$

The above representation has the advantage that for the Gaussian fixed point, $A_k^* = 0$ for all $k \geq 1$, and as a result the linearised RG transformation $\boldsymbol{V}'$ is diagonal. Thus from (4.30), since from (4.28) $\lambda'^* = 1$, the eigenvalues of $\boldsymbol{V}'$ are

$$\Lambda_k = 2c^{-k} = 2^{(\sigma-1)k+1}, \qquad k = 1, 2, \ldots. \tag{4.31}$$

When $0 < \sigma < 1/2$ only the first eigenvalue $\Lambda_1 = 2^{\sigma}$ exceeds unity as required and (4.20) and (4.21) yield the 'classical' exponents

$$\nu = 1/\sigma, \qquad \gamma = 1, \text{etc} \tag{4.32}$$

in agreement with previous results. In the range $1/2 \leq \sigma < 1$ we obtain a cascade of bifurcations as successive eigenvalues pass through unity, i.e. at points

$$\sigma = 1 - k^{-1}, \qquad k = 2, 3, \ldots. \tag{4.33}$$

It follows that in this range the Gaussian fixed point cannot be reached by iteration from $\tilde{P}_0(x)$ by adjusting only the one physical parameter β. It may, however, correspond to a higher-order critical point for a different model $\tilde{P}_0(x)$ containing more than one adjustable parameter (Riedel and Wegner 1972, Baker and Golner 1977).

Before discussing our numerical procedure and results, particularly in the range $1/2 < \sigma < 1$, we note that the borderline case $\sigma = 1/2$ requires special attention since in this case the second eigenvalue Λ_2 is unity and hence the Gaussian fixed point is only 'marginally stable'. We need then to consider the non-linear effect of the RGE (4.27)–(4.28) which in general is expected (Wegner and Riedel 1973) to give rise to fractional powers of $\ln|\beta-\beta_c|$, in the critical behaviour of the system. To see this we follow Wegner and Riedel (1973) and approximate (4.27)–(4.28) by neglecting all coefficients except $A_1^{(l)}$ and $A_2^{(l)}$.

In this way we obtain, when $\sigma=1/2$

$$A_1^{(l+1)}\approx 2^{1/2}A_1^{(l)}(1+48A_2^{(l)}) \tag{4.34}$$

and

$$A_2^{(l+1)}\approx A_2^{(l)}(1+144A_2^{(l)}) \tag{4.35}$$

where $T_{1,1,2}$ and $T_{2,2,2}$ have been written explicitly. From (4.35) we obtain

$$A_2^{(l)}\approx -(1/144)(l+l_0)^{-1} \qquad \text{as } l\to\infty. \tag{4.36}$$

Substituting (4.36) and the *ansatz*

$$A_1^{(l)}\sim A\,2^{l/2}(l+l_0)^p \tag{4.37}$$

into (4.34) we find that

$$A_1^{(l)}\sim 2^{(l+l_0)/2}(l+l_0)^{-1/3}. \tag{4.38}$$

Then, following the same line of argument as in Fisher *et al* (1972), leads us to the asymptotic behaviour of the correlation length ξ and the susceptibility χ in the neighbourhood of the critical point:

$$\xi\sim t^{-2}(\ln t^{-1})^{2/3} \tag{4.39}$$

and

$$\chi\sim t^{-1}(\ln t^{-1})^{1/3} \tag{4.40}$$

as $t=|\beta-\beta_c|\to 0$. These results are in agreement with those of Fisher *et al* (1972) for the power-law potential Ising model.

In order to examine our procedure, involving the RGE (4.10)–(4.12), and the linearised RGE (4.14), numerically, we need to be able to terminate the infinite sums appearing in these equations at some finite value $k=M$. If the magnitude of the fixed point B_k^* decreases sufficiently rapidly as k increases, the effect of terminating the sums should be small, and the approximation, which we will call the M-approximation, should improve with increasing M. In fact it will be seen that all numerical results are practically the same in the M-approximation for $M\geqslant 10$.

In the M-approximation, the critical temperature β_c is the value of β in (4.19) for which $B_k^{(l)}\to B_k^*$, $k=1,2,\ldots,M$, as $l\to\infty$. Typical behaviour of the $B_k^{(l)}$ as a function of l is shown schematically in figure 1 for several values of β and $k=1$. From this behaviour we were motivated to take the following practical scheme to determine β_c: Let $\{\beta_l\}$ be the sequence of values of β for which

$$B_1^{(l-1)}(\beta_l)=B_1^{(l)}(\beta_l). \tag{4.41}$$

Then it is reasonable to expect that the sequence of β_l will approach the limit β_c as $l\to\infty$. Such a sequence of β_l for $\sigma=1/4$ in the 12-approximation is shown in table 1. The sequence, for sufficiently large l, converges with approximately exponential behaviour. Thus, we can form a new sequence $\{(\beta_l'\}$ by an exponential interpolation using β_{l-2}, β_{l-1} and β_l. That is, we fit β_l to the form

$$\beta_l=\beta_c+\text{constant}\times b^{-l} \tag{4.42}$$

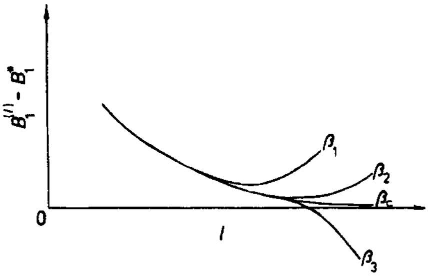

Figure 1. Schematic behaviour of $B_1^{(l)}$ against l for temperatures close to β_c. Here $\beta_1>\beta_2>\beta_c>\beta_3$.

Table 1. A portion of the β_l and β_l' sequences for $\sigma = 1/4$ in the 12-approximation.

l	$(\beta_l - 0{\cdot}116028)\times 10^{10}$	$(\beta_l' - 0{\cdot}116028)\times 10^{10}$
25	10 120	9768
26	9976	9762
27	9891	9769
28	9840	9764
29	9810	9767
30	9792	9765
31	9781	9764

and obtain β_l' from the value of β_c determined from β_{l-2}, β_{l-1}, β_l and (4.42). This new sequence is also shown in table 1 from which we conclude that $\beta_c \simeq 0{\cdot}11602898$ when $\sigma = 1/4$.

Having determined β_c, the next task is to find Λ_1. One way of doing this is to use the values of β_c, obtained by the above method, in (4.19) and generate the $B_k^{(l)}$, which for large l, should approach the fixed point values B_k^* to be used in (4.15) and (4.16). This procedure, however, has the disadvantage that it is very sensitive to small errors in β_c. As an alternative method we first look at the $B_k^{(l)}$ at $\beta = \beta_l$ where β_l satisfies (4.41). In table 2 we show $B_1^{(l)}(\beta_l)$ and $B_2^{(l)}(\beta_l)$ for a range of l, for $\sigma = 1/4$ in the 12-approximation. These sequences also converge very rapidly and if we take the 3-point extrapolation as in (4.42) we obtain the values in parentheses, which are in excellent

Table 2. A portion of the $B_1^{(l)}(\beta_l)$ and $B_2^{(l)}(\beta_l)$ sequences and the corresponding 3-point extrapolations (in parentheses) for $\sigma = 1/4$ in the 12-approximation.

l	$(B_1^{(l)}(\beta_l) - 3{\cdot}977\times 10^{-2})\times 10^9$		$(B_2^{(l)}(\beta_l) - 7{\cdot}910\times 10^{-4})\times 10^{11}$	
25	6992	(5906)	4162	(6089)
26	6671	(5890)	4728	(6091)
27	6444	(5896)	5129	(6104)
28	6283	(5890)	5412	(6091)
29	6170	(5904)	5612	(6094)
30	6090	(5896)	5754	(6102)
31	6033	(5892)	5854	(6092)
Exact value for the Gaussian fixed point		5896		6096

agreement with the analytic result (4.23) listed at the bottom of table 2. Here we see, as expected, that for $\sigma = 1/4$, the fixed point is indeed the Gaussian.

Encouraged by this success we took $B_k^{(l)}(\beta_l)$, $k = 1, 2, \ldots, M$ as the lth approximation to B_k^* and used these values in (4.15) to find the relevant eigenvalue which we denote by $\Lambda_{1,l}$. Since we are only interested in the one eigenvalue exceeding unity, the simplest way we could find to determine that eigenvalue was to use Newton iteration to find $\Lambda_1 > 1$ for which $\det(\boldsymbol{V} - \Lambda_1 \boldsymbol{I}) = 0$. The determinants of the $M \times M$ matrices were calculated by the Gaussian elimination method and all algorithms were set up in the double precision mode. A sequence of $\Lambda_{1,l}$ thus obtained for $\sigma = 1/4$ in the 12-approximation is shown in table 3 together with the 3-point extrapolations. From this table one would conclude that $\Lambda_1 \simeq 1{\cdot}892071$ whereas the exact value for the Gaussian fixed point is $\Lambda_1 = 1{\cdot}189207115$.

Table 3. A portion of the $\Lambda_{1,l}$ sequence and the 3-point extrapolated values for $\sigma = 1/4$ in the 12-approximation.

l	$\Lambda_{1,l}$	Extrapolated values
25	1·189211234	1·18920711
26	1·189210028	1·18920713
27	1·189209174	1·18920710
28	1·189208571	1·18920712
29	1·189208145	1·18920712
30	1·189207843	1·18920711
31	1·189207630	1·18920712
Exact value for the Gaussian fixed point		$2^{\sigma} = 1{\cdot}189207115$

The above algorithm proved successful over the whole range $0 < \sigma < 1$ except near $\sigma = 1/2$ where the convergence of $\Lambda_{1,l}$ was very slow. We found that the convergence of β_l and $\Lambda_{1,l}$ for $\sigma \geqslant 1/2$ could be improved by generating β_l for which

$$B_2^{(l-1)}(\beta_l) = B_2^{(l)}(\beta_l) \tag{4.44}$$

instead of (4.41). Even so, at $\sigma = 0{\cdot}55$ for example, we needed sequences up to $l \simeq 46$ to obtain four-digit accuracy for Λ_1, whereas for values of σ close to unity, $\Lambda_{1,l}$ sequences up to $l = 20$ were sufficient to obtain eight-digit accuracy.

In table 4 we show how the $\Lambda_{1,l}$ change with M for three values of σ. As mentioned before, for $M \geqslant 10$ there is essentially no change in the $\Lambda_{1,l}$ to the accuracy indicated in the tables. It was sufficient then for our purposes to fix $M = 12$. The critical tempera-

Table 4. Dependence of Λ_1 on the size of the approximation. M is the value at which the infinite sums in (4.11), (4.12) and (4.15) are cut off.

M	$\sigma = 1/4$	$\sigma = 0{\cdot}55$	$\sigma = 3/4$
4	1·188944	1·3936	1·39211
6	1·189205	1·4204	1·40430
8	1·189207	1·425	1·40449
10	1·189207	1·426	1·40449
12	1·189207	1·426	1·40449

tures and Λ_1 thus obtained for a range of $0<\sigma<1$ are summarised in table 5. For $\sigma \leq 0{\cdot}35$ we confirmed that the fixed points are all Gaussian to the accuracy indicated in tables 2 and 3. As σ approached one-half the convergence was slow and the accuracy of the approximation was consequently decreased. There is little doubt, however, that the fixed points are all Gaussian for $0<\sigma<1/2$. As σ increases from one-half, $\Lambda_1(\sigma)$ at first increases slowly then decreases until it reaches the value unity at $\sigma = 1$, where the critical temperature is zero and the phase transition vanishes. The behaviour in the neighbourhood of $\sigma = 1$ is discussed in § 6.

Table 5. The values of β_c, Λ_1, ν and γ for $0<\sigma<1$. The errors are at most ±1 in the last digit shown.

σ	β_c	Λ_1	ν	γ
0·05	0·01827630			
0·10	0·03864942			
0·15	0·06145812			
0·20	0·08709735			
0·25	0·11602898	2^σ	$1/\sigma$	1
0·30	0·14879440			
0·35	0·18602929			
0·40	0·22848165			
0·45	0·27703568			
0·50	0·33274779			
0·55	0·39690639	1·426	1·953	1·074
0·60	0·47113154	1·4306	1·936	1·161
0·65	0·55753771	1·4291	1·941	1·262
0·70	0·65900811	1·42087	1·973	1·381
0·75	0·77969366	1·40449	2·041	1·530
0·80	0·92602204	1·37769	2·163	1·731
0·85	1·1090596	1·33668	2·389	2·030
0·90	1·3514449	1·27464	2·856	2·571
0·95	1·7185339	1·17718	4·249	4·037
0·98	2·1285530	1·087516	8·262	8·097
0·99	2·4046601	1·048558	14·62	14·47
0·995	2·6673358	1·026371	26·63	26·50
0·999	3·2628324	1·006219	$1{\cdot}118\times10^2$	$1{\cdot}117\times10^2$
0·9998	3·8587028	1·001445	$4{\cdot}80\times10^2$	$4{\cdot}80\times10^2$

Finally, one might wonder why we used the B_k rather than the A_k: (4.25)–(4.30). The whole success of our numerical scheme relies on the Hermite coefficients decreasing sufficiently rapidly for the class of functions considered, so that the termination of the infinite sums at a finite value is justified numerically. This is not the case for the A_k and numerical experiments with $A_k^{(l)}$ proved useless for $\sigma > 1/2$.

5. $\Delta\sigma$-expansion

For $1/2<\sigma<1$, a non-Gaussian fixed point, which we have been unable to find analytically, determines the critical behaviour of the HM. For σ close to and exceeding one-half, however, an expansion of the maximum eigenvalue Λ_1 of $\boldsymbol{V}'$ equation (4.30) in powers of $\Delta\sigma = \sigma - 1/2$ can easily be constructed. In the usual RG analysis (Wilson and Fisher 1972) this expansion corresponds to the well known $\epsilon = 4 - d$ expansion.

From (4.27) we see that when

$$\sigma = 1/2 + \Delta\sigma \equiv 1/2 + \epsilon/\ln 2 \tag{5.1}$$

a non-Gaussian solution arises for which $A_2^* = -\epsilon/72 + O(\epsilon^2)$. Since $T_{k,2,2} = 0$ for $k > 4$ we then immediately see that A_1^*, A_3^* and A_4^* are of order ϵ^2 and since $T_{k,1,2} = T_{k,3,2} = T_{k,4,2} = 0$ for $k > 6$, only A_5^* and A_6^* are of order ϵ^3. In general, due to the nature of the $T_{k,k',k''}$, A_{2k-1}^* and A_{2k}^* are of order ϵ^k except when $k = 1$.

For this non-Gaussian fixed point, (4.30) becomes to first order in ϵ,

$$V'_{k,k'} = 2^{1-k/2}[\delta_{kk'} + (k\delta_{kk'} - T_{k,k',2}/72)\epsilon + O(\epsilon^2)] \tag{5.2}$$

with eigenvalues

$$\Lambda_k = 2^{1-k/2}\left\{1 + \left[k - \frac{2}{3}\binom{2k}{2}\right]\epsilon + O(\epsilon^2)\right\}. \tag{5.3}$$

For $k = 2$, $\Lambda_2 = 1 - 2\epsilon + O(\epsilon^2)$ so that provided $\epsilon > 0$ (i.e. $\sigma > 1/2$), only Λ_1 exceeds unity for sufficiently small ϵ, as required of the physical fixed point.

We have carried through the straightforward and infinitely tedious perturbation calculation to third order in ϵ and find that

$$\begin{aligned}
A_1^* &= -2(2+\surd 2)\epsilon^2/27 - 14(4+3\surd 2)\epsilon^3/27 + O(\epsilon^4)\\
A_2^* &= -\epsilon 72 - (15+16\surd 2)\epsilon^2/216 + (209+144\surd 2)\epsilon^3/324 + O(\epsilon^4)\\
A_3^* &= (1+\surd 2)\epsilon^2/324 + (5+3\surd 2)\epsilon^3/162 + O(\epsilon^4)\\
A_4^* &= \epsilon^2/10368 - (33+16\surd 2)\epsilon^3/15552 + O(\epsilon^4)\\
A_5^* &= -(1+\surd 2)\epsilon^3/23328 + O(\epsilon^4)\\
A_6^* &= -\epsilon^3/2239488 + O(\epsilon^4)
\end{aligned} \tag{5.4}$$

with all other A_k^* of higher order in ϵ than ϵ^3. The maximum eigenvalue of $\boldsymbol{V}$ was then found to be

$$\Lambda_1 = (\surd 2)[1 + \epsilon/3 - (23+32\surd 2)\epsilon^2/18 + (1699+1152\surd 2)\epsilon^3/54 + O(\epsilon^4)]. \tag{5.5}$$

Blekher and Sinai (1974) on the other hand report that, in our notation,

$$\Lambda_1 = (\surd 2)[1 + \epsilon/3 + (71+130\surd 2)\epsilon^2/54 + O(\epsilon^3)] \tag{5.6}$$

which differs from our result in the second-order term. Numerically for $\sigma = 0{\cdot}55$, corresponding to $\epsilon = 0{\cdot}05 \ln 2$, we find in § 4 that $\Lambda_1 = 1{\cdot}426$. The partial sums to second- and third-order in (5.5) for this value of ϵ give 1·424 and 1·428 respectively, which is in very good agreement with the numerical value. The partial sum to second-order from (5.6) on the other hand gives 1·439 so we are inclined to believe our result (5.5) especially as we have checked it independently on several occasions.

For the critical exponents ν and γ, (5.5), (4.20) and (4.21) yield, in terms of $\Delta\sigma$,

$$\begin{aligned}
\nu^{-1} = \tfrac{1}{2}[1 &+ 2\Delta\sigma/3 - 8(3+4\surd 2)(\Delta\sigma)^3(\ln 2)/9\\
&+ 16(323+222\surd 2)(\Delta\sigma)^3(\ln 2)^2/81 + O(\Delta\sigma)^4]
\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}
\gamma = 1 + 4\Delta\sigma/3 &+ 8[(3+4\surd 2)(\ln 2) - 1](\Delta\sigma)^2/9 - 16[(323+222\surd 2)(\ln 2)^2\\
&- 3(3+4\surd 2)\ln 2 - 3](\Delta\sigma)^3/81 + O(\Delta\sigma)^4.
\end{aligned} \tag{5.8}$$

Equation (5.8) is to be compared with the result for the Ising model with a power-law potential obtained by Fisher *et al* (1972):

$$\gamma = 1 + 4\Delta\sigma/3 + 8(4\ln 2 + \pi - 1)(\Delta\sigma)^2/9 + O(\Delta\sigma)^3. \tag{5.9}$$

The critical exponents for the HM and Ising model with power potentials are then seen to agree to first-order in $\Delta\sigma$ but differ slightly to second-order. Numerically, the coefficients of $(\Delta\sigma)^2$ are 4·445 and 4·368 for (5.8) and (5.9) respectively.

It is apparent that all of the above expansions are asymptotic but nevertheless, they should give fairly accurate estimates for $\Delta\sigma \leqslant 0{\cdot}05$.

6. Discussion and summary

In this paper we have investigated Dyson's hierarchical model in one-dimension, with 'potential' falling off like $r^{-(1+\sigma)}$, in the range $0<\sigma<1$ where a phase transition is known to occur. A new exact renormalisation group recursion relation, which is ideally suited to numerical and analytical investigation, was derived for a 'dual' spin probability density, giving 'classical' critical exponents for $0<\sigma\leqslant 1/2$ and non-classical exponents for $1/2<\sigma<1$. A scaling type argument gave $\eta = 2-\sigma$ and $\delta = (1+\sigma)/(1-\sigma)$ for all $0<\sigma<1$. While the former is probably true, independent series analysis suggests that δ sticks at its classical value of three for $\sigma \leqslant 1/2$.

In the non-classical region $1/2<\sigma<1$ we obtained accurate numerical values for the critical exponents ν and γ, given in table 5. Other critical exponents can then be obtained from the usual scaling laws. We also obtained various $\Delta\sigma = \sigma - 1/2$ expansions, analogous to the usual RG $\epsilon = 4-d$ expansions, to third-order in $\Delta\sigma$, which are in close agreement with our numerical values for small $\Delta\sigma$ (see figure 2) but disagree with similar expansions obtained to second-order in $\Delta\sigma$ by Blekher and Sinai (1974). Our numerical results are, however, in qualitative agreement with similar results obtained by Blekher (Blekher and Sinai 1975) and most recently by Baker and Golner (1977).

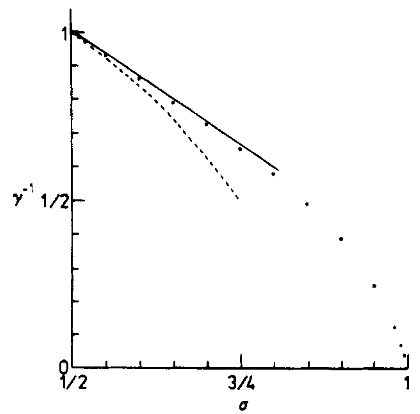

Figure 2. γ^{-1} against σ for $1/2\leqslant\sigma<1$ obtained from table 5 (full circles). The full and broken lines are the $\Delta\sigma$-expansions of γ^{-1} truncated at first- and second-order respectively in $\Delta\sigma$. For $0<\sigma\leqslant 1/2$, $\gamma = 1$.

We were not successful in finding a simple formula to fit our numerical values for $1/2<\sigma<1$ to the accuracy indicated in table 5. The best 'simple expression' we were able to concoct was

$$\ln \Lambda_1 \simeq (1-\sigma)\ln(1-\sigma)^{-1} \tag{6.1}$$

which differs from the values given in table 5 by less than $1\frac{1}{2}\%$ for $1/2 \leqslant \sigma < 1$. For σ close to unity ($\sigma \geqslant 0.99$) a better approximation is

$$\ln \Lambda_1 \simeq 0.64(1-\sigma)\ln(1-\sigma)^{-1} + 1.8(1-\sigma). \tag{6.2}$$

In any event, it seems clear that $\ln \Lambda_1$, and hence the critical exponents ν, γ, etc. are non-analytic at $\sigma = 1$ as well as at $\sigma = 1/2$ as suggested in § 5.

Granted that $\ln \Lambda_1$ approaches zero as σ approaches unity, the critical exponents γ and ν become infinite as σ approaches unity. The critical exponent β on the other hand, given by

$$\beta = (1-\sigma)\ln 2/2\ln \Lambda_1 \tag{6.3}$$

for $1/2<\sigma<1$ decreases to zero as σ approaches unity. This behaviour, together with the fact that the critical temperature approaches zero as σ approaches unity, seems to suggest that the HM exhibits essentially singular behaviour at zero temperature and σ equal to unity, in accordance with the one-dimensional short-range interaction Ising model.

In figure 2 we have plotted the values of γ^{-1} against σ in the range $1/2<\sigma<1$. It is to be noted here that γ is monotonic in σ even though $\ln \Lambda_1$, and hence ν, is not. Also, as seen from figure 2, the γ^{-1} data fall very nearly on a straight line for $1/2 \leqslant \sigma \leqslant 3/4$, so that in this range, $\log \Lambda_1$ can also be approximated by

$$\ln \Lambda_1 \simeq \sigma(1.175 - 0.964\sigma). \tag{6.4}$$

Also shown on figure 2 are the partial sums to first- and second-order in $\Delta\sigma$ of the $\Delta\sigma$-expansion for γ^{-1}, which from (5.8) reads

$$\gamma^{-1} = 1 - 4\Delta\sigma/3 - 8[(3+4\surd 2)\ln 2 - 3](\Delta\sigma)^2/9 + O(\Delta\sigma)^3. \tag{6.5}$$

The partial sums to third-order diverge more strongly than the second-order sums as $\Delta\sigma$ increases.

Finally, when we compare the behaviour of the critical exponents of the HM for σ close to unity with that of the corresponding long-range Ising model, obtained numerically by Nagle and Bonner (1970), we find that, except for γ, there is rough qualitative agreement. As σ approaches unity we find that γ diverges for the HM whereas Nagle and Bonner report $\gamma \simeq 2.2$ for the Ising model. This discrepancy is not altogether surprising in view of the fact that the HM has no phase transition at $\sigma = 1$, while for the Ising model, there is a strong possibility (Thouless 1969) that there is a phase transition when $\sigma = 1$, corresponding to an inverse square law potential. The critical behaviour of the two models is, on the other hand, identical for $0<\sigma \leqslant 1/2$ and 'almost identical' for $\sigma \geqslant 1/2$ as indicated in §§ 4 and 5 respectively. It is quite likely that the two models have different critical behaviour for $1/2<\sigma<1$; becoming more apparent as σ approaches unity. The lack of accurate numerical data for the Ising model at this stage, however, makes it impossible to assess this difference in any quantitative way.

Acknowledgment

We are grateful to the Australian Research Grants Commission for its support.

Appendix. Derivation of equations (2.17) and (2.18)

Let $S_{p,r}$, $p=0,\ldots,N$, $r=1,\ldots,2^{N-p}$ have the same hierarchical meaning as in equation (2.2) and let us define a $2^N\times 2^N$ symmetric matrix J whose elements are given through

$$\sum_{i=1}^{2^N}\sum_{j=1}^{2^N} S_{0,i}J_{i,j}S_{0,j}=\sum_{p=0}^{N} 2^{-2p}b_p\sum_{r=1}^{2^{N-p}}(S_{p,r})^2 \tag{A.1}$$

where b_p are arbitrary constants. The structure of such matrices was investigated by McGuire (1973). In particular, the eigenvectors of J are independent of b_p and the eigenvalues are

$$\lambda_k=\sum_{p=0}^{k} 2^{-p}b_p, \qquad \text{for } k=0,1,2,\ldots,N, \tag{A.2}$$

λ_k being 2^{N-k-1}-fold degenerate for $k=0,1,\ldots,N-1$, and nondegenerate for $k=N$. Also,

$$\sum_{j=1}^{2^N} J_{ij}=\lambda_N \qquad \text{for any } i. \tag{A.3}$$

Since the eigenvectors of J are independent of b_p, if we consider

$$\sum_i\sum_j S_{0,i}J'_{ij}S_{0,j}=\sum_{p=0}^{N} 2^{-2p}b'_p\sum_{r=1}^{2^{N-p}}(S_{p,r})^2 \tag{A.4}$$

with b'_p chosen so as to satisfy

$$\left(\sum_{p=0}^{k} 2^{-p}b'_p\right)\left(\sum_{p=0}^{k} 2^{-p}b_p\right)=1 \tag{A.5}$$

for all $k=0,1,\ldots,N$, then it is not difficult to see that J' is the inverse of J.

Now to derive equation (2.17), we first write equation (2.12) in the form

$$Q_N^{(l)}(\beta,H)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(\beta\sum_{p=0}^{N}2^{-2p}c^p\sum_{r=1}^{2^{N-p}}S_{p,r}^2+\beta H\sum_{r=1}^{2^N}S_{0,r}\right)$$
$$\times\prod_{r=1}^{2^N}\exp(-\beta S_{0,r}^2)P_l(S_{0,r})\,\mathrm{d}S_{0,r} \tag{A.6}$$

where we have added and subtracted a term $\beta\sum_{r=1}^{2^N}S_{0,r}^2$ in the exponent.

Identifying $b_p=c^p$ in equation (A.1), we write (A.6) as

$$Q_N^{(l)}(\beta,H)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(\beta\sum_{r,r'}S_{0,r}J_{r,r'}S_{0,r'}+\beta H\sum_r S_{0,r}\right)$$
$$\times\prod_{r=1}^{2^N}\exp(-\beta S_{0,r}^2)P_l(S_{0,r})\,\mathrm{d}S_{0,r}. \tag{A.7}$$

Next, we utilise the familiar identity

$$\exp\Big(\beta \sum_{i,j} x_i J_{ij} x_j\Big)$$
$$= (4\beta\pi)^{-2^{N-1}} D^{-1/2} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\Big(-(1/4\beta)\sum_{i,j} y_i J_{i,j}^{-1} y_j + \sum_i y_i x_i\Big) \prod_i \mathrm{d}y_i, \tag{A.8}$$

D being the determinant of J, to change equation (A.7) to

$$Q_N^{(l)}(\beta, H) = (4\beta\pi)^{-2^{N-1}} D^{-1/2} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\Big(-(1/4\beta)\sum_{i,j} y_i J_{i,j}^{-1} y_j\Big)$$
$$\times \prod_{i=1}^{2N} \Big(\int_{-\infty}^{\infty} \exp(y_i x - \beta x^2 + \beta H x) P_l(x)\,\mathrm{d}x\Big)\,\mathrm{d}y_i. \tag{A.9}$$

If we change the integration variable in (A.9) to

$$z_i = \beta^{-1/2}(y_i + \beta H)/2, \tag{A.10}$$

equation (A.9) becomes

$$Q_N^{(l)}(\beta, H) = \pi^{-2^{N-1}} D^{-1/2} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\Big(-\sum_{i,j} J_{i,j}^{-1}(z_i - \beta^{1/2}H/2)(z_j - \beta^{1/2}H/2\Big)$$
$$\times \prod_{i=1}^{2N} \Big(\int_{-\infty}^{\infty} \exp(-\beta x^2 + 2\beta^{1/2} z_i x) P_l(x)\,\mathrm{d}x\Big)\,\mathrm{d}z_i$$
$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp(-\beta \mathcal{H}_N') \prod_{r=1}^{2N} \tilde{P}_l(z_r)\,\mathrm{d}z_r \tag{A.11}$$

where $\tilde{P}_l(x)$ is related to $P_l(x)$ by (2.18) and $\mathcal{H}_N'$ is defined by

$$\beta\mathcal{H}_N' = \sum_{i,j} J_{i,j}^{-1} z_i z_j - (\beta^{1/2}H/2) \sum_{i,j} (J_{i,j}^{-1} + J_{j,i}^{-1}) z_i + (\beta H^2/4) \sum_{i,j} J_{i,j}^{-1} + 2^{N-1} \ln \pi + (\tfrac{1}{2}) \ln D. \tag{A.12}$$

Equation (A.12) is the required equation (2.17). The precise form of $\mathcal{H}_N'$ is not important here. We merely note that from (A.4) ($J' = J^{-1}$), $\mathcal{H}_N'$ has the same hierarchical structure as the original $\mathcal{H}_N$.

To complete the logical circle we need to show that $\tilde{P}_l(x)$ given in terms of $P_l(x)$ by (2.18) satisfies the recursion relation (2.8), and is in fact related to the partition function by (2.7).

Firstly from (2.18) and (2.14) we obtain

$$\tilde{P}_{l+1}(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(2\beta^{1/2} x y) 2c^{-1/2} P_l(c^{-1/2}x + z) P_l(c^{-1/2}x - z)\,\mathrm{d}z\,\mathrm{d}x. \tag{A.13}$$

Changing variables to $u = c^{-1/2}x + z$ and $v = c^{-1/2}x - z$ then gives

$$\tilde{P}_{l+1}(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[(\beta c)^{1/2} y(u+v)] P_l(u) P_l(v)\,\mathrm{d}u\,\mathrm{d}v$$
$$= \Big(\int_{-\infty}^{\infty} \exp[(\beta c)^{1/2} y u] P_l(u)\,\mathrm{d}u\Big)^2. \tag{A.14}$$

On the other hand, from (2.18)

$$\pi^{-1/2}\int_{-\infty}^{\infty} \tilde{P}_l(x)\exp[-(x-y)^2]\,\mathrm{d}x$$
$$=\pi^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-(x-y)^2]\exp(-\beta u^2+2\beta^{1/2}ux)P_l(u)\,\mathrm{d}u\,\mathrm{d}x. \qquad \text{(A.15)}$$

Interchanging the order of integration and integrating on x then gives

$$\pi^{-1/2}\int_{-\infty}^{\infty} \exp[-(x-y)^2]\tilde{P}_l(x)\,\mathrm{d}x=\int_{-\infty}^{\infty} \exp(2\beta^{1/2}yu)P_l(u)\,\mathrm{d}u. \qquad \text{(A.16)}$$

Comparing (A.14) and (A.16) gives the required recursion relation (2.8).

Finally from (2.19) and (2.8) it is easily shown that (2.6) is valid for $l=0$. The fact that it is valid for all $l=1,2,\ldots$ then follows easily by induction.

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